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A REMARK ON THE PROJECTION-3 METHOD

JIE SHEN

Department of Mathematics, Penn State University, University Park, PA 16802, U.S.A.

SUMMARY

We show that the continuous (in time) form of the projection-3 scheme proposed in Reference 2 is not a proper approximation of the unsteady Navier–Stokes equations. Hence, the projection-3 scheme and its variants are not appropriate for the numerical computation of the Navier–Stokes equations.

KEY WORDS Navier–Stokes equations Projection method Time discretization

The projection-3 scheme[1] was proposed as a possible improvement over the projection-1 and projection-2 schemes.[1] To understand better the nature of these projection schemes, we will first exploit an intrinsic relation between the three schemes.[1,2] The classical projection method for solving the unsteady Navier–Stokes equations

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0, \quad \in \Omega\times\mathbb{R}^+, \tag{1}$$

was initially proposed by Chorin[3] and Temam.[4] A semi-discretized version (named as projection-1 scheme[1]) of the classical projection method applied to the Navier–Stokes equations with homogeneous Dirichlet boundary condition can be written as follows:

let $\mathbf{u}^0 = \mathbf{u}_0$; we solve successively $\tilde{\mathbf{u}}^{n+1}$ and $\{\mathbf{u}^{n+1}, p^{n+1}\}$ by

$$\frac{(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n)}{\Delta t} - \nu\Delta\tilde{\mathbf{u}}^{n+1} + (\tilde{\mathbf{u}}^{n+1}\cdot\nabla)\tilde{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1} \quad \text{in } \Omega,\ \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0, \tag{2}$$

$$\frac{(\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1})}{\Delta t} + \nabla p^{n+1} = 0, \quad \operatorname{div}\mathbf{u}^{n+1} = 0, \text{ in } \Omega,\ \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{3}$$

It is well known that the above scheme suffers from large splitting errors. A number of modified projection schemes have been proposed to improve the accuracy (see, for instance, Reference 5 for a review). A typical modified scheme is:

$$\frac{(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n)}{\Delta t} - \nu\Delta\tilde{\mathbf{u}}^{n+1} + (\tilde{\mathbf{u}}^{n+1}\cdot\nabla)\tilde{\mathbf{u}}^{n+1} + \nabla p^n = \mathbf{f}^{n+1} \quad \text{in } \Omega,\ \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0, \tag{4}$$

$$\frac{(\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1})}{\Delta t} + \tfrac{1}{2}\nabla(p^{n+1} - p^n) = 0, \quad \operatorname{div}\mathbf{u}^{n+1} = 0 \quad \text{in } \Omega,\ \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{5}$$

One can find in Reference 6, a detailed error analysis for both schemes.

The projection-2 scheme[1] is, in fact, a scheme similar to (4) and (5) in the sense that the Crank–Nicolson treatment was used in favour of backward Euler in (4). We refer to Reference 5 for more related schemes.

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It is clear that the above schemes can also be viewed as (Stokes) operator-splitting schemes, and that the non-linear term does not play any essential role here. Hence, to simplify our presentation, we shall focus on the linearized (dropping the non-linear term) Navier–Stokes equations.

We note that $\{\mathbf{u}^n\}$ in (2–3) and (4–5) can be eliminated to form systems only involving $\{\tilde{\mathbf{u}}^n\}$. In fact, taking the sum of (2) at step n and (3) at step $n-1$ and applying the divergence operator to (3), after dropping the non-linear term, we find that (2) and (3) are equivalent to

$$\frac{(\tilde{\mathbf{u}}^{n+1}-\tilde{\mathbf{u}}^n)}{\Delta t}-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\nabla p^n=\mathbf{f}^{n+1}, \tag{6}$$

$$\operatorname{div}\tilde{\mathbf{u}}^{n+1}-\Delta t\Delta p^{n+1}=0,\quad \left.\frac{\partial p^{n+1}}{\partial\mathbf{n}}\right|_{\partial\Omega}=0. \tag{7}$$

Similarly, (4) and (5) are equivalent to

$$\frac{(\tilde{\mathbf{u}}^{n+1}-\tilde{\mathbf{u}}^n)}{\Delta t}-\nu\Delta\tilde{\mathbf{u}}^{n+1}+\tfrac{1}{2}\nabla(3p^n-p^{n-1})=\mathbf{f}^{n+1}, \tag{8}$$

$$\operatorname{div}\tilde{\mathbf{u}}^{n+1}-\frac{\Delta t}{2}\Delta(p^{n+1}-p^n)=0,\quad \left.\frac{\partial p^{n+1}}{\partial\mathbf{n}}\right|_{\partial\Omega}=\left.\frac{\partial p^{n}}{\partial\mathbf{n}}\right|_{\partial\Omega}. \tag{9}$$

We note that (6) (resp. (8)) can be replaced by higher-order time discretization schemes so that the leading error term of the schemes is dictated by the truncation error introduced by (7) (resp. (9)). But, unfortunately, any attempt to reduce this error by replacing Δt in (7) and (9) with $(\Delta t)^\alpha$ for any $\alpha>1$ would result in an unstable scheme.

To understand why (8) and (9) are superior than (6) and (7), let us write down the continuous (in time) forms of the above schemes. For (6) and (7), the continuous form is:

$$\mathbf{u}_t^\varepsilon-\Delta\mathbf{u}^\varepsilon+\nabla p^\varepsilon=\mathbf{f}, \tag{10}$$

$$\operatorname{div}\mathbf{u}^\varepsilon-\varepsilon\Delta p^\varepsilon=0,\quad \left.\frac{\partial p^\varepsilon}{\partial\mathbf{n}}\right|_{\partial\Omega}=0, \tag{11}$$

with $\varepsilon=\Delta t$. On the other hand, the continuous form for (8) and (9) is:

$$\mathbf{u}_t^\varepsilon-\Delta\mathbf{u}^\varepsilon+\nabla p^\varepsilon=\mathbf{f}, \tag{12}$$

$$\operatorname{div}\mathbf{u}^\varepsilon-\varepsilon\Delta p_t^\varepsilon=0,\quad \left.\frac{\partial p_t^\varepsilon}{\partial\mathbf{n}}\right|_{\partial\Omega}=0, \tag{13}$$

with $\varepsilon=\frac{1}{2}(\Delta t)^2$. It is shown from Reference 7 that, for (10) and (11), we have the error estimate

$$\|\mathbf{u}(t)-\mathbf{u}^\varepsilon(t)\|_{L^2(\Omega)}\sim O(\varepsilon). \tag{14}$$

It can be shown from Reference 4 that under suitable assumptions the same estimate holds for (12) and (13). One then realize that the improvement of (12) and (13) over (10) and (11) comes from the fact that by replacing Δp in (11) by Δp_t, we were able to choose $\varepsilon=\frac{1}{2}(\Delta t)^2$ in (13). We then realize that a second-order time stepping scheme for (12) and (13) would result in a second-order projection scheme. In fact, it has recently been shown[4] that the projection-2 scheme with Crank–Nicolson time stepping is fully second-order accurate in time. Therefore, by intuition, one would naturally suggest that the formulation

$$\mathbf{u}_t^\varepsilon-\Delta\mathbf{u}^\varepsilon+\nabla p^\varepsilon=\mathbf{f}, \tag{15}$$

$$\operatorname{div}\mathbf{u}^\varepsilon-\varepsilon\Delta p_{tt}^\varepsilon=0,\quad \left.\frac{\partial p_{tt}^\varepsilon}{\partial\mathbf{n}}\right|_{\partial\Omega}=0, \tag{16}$$

would lead to an improved scheme over (4) and (5), since we can choose $\varepsilon \sim (\Delta t)^3$ in a time-discretized scheme of (15) and (16).

In fact, after similar reformulation as before, we find that the projection-3 scheme[1] can be viewed as a discrete version of (15) and (16). But, unfortunately, as proved below, the formulation (15) and (16) is not a proper approximation of the linearized Navier–Stokes equations. Hence, no scheme based on the discretization of (15) and (16) will be appropriate for the approximation of the Navier–Stokes equations.

We now show that the solution of (15) and (16) cannot be bounded uniformly for $t \in [\delta, +\infty)$ for any $\delta > 0$. While, on the contrary, for any $\delta > 0$, the solution of the (linearized) Navier–Stokes equations is uniformly bounded for $t \in [\delta, +\infty)$.

Applying the divergence operator to (15) and taking into account (16), we derive

$$\varepsilon \Delta p^{\varepsilon}_{ttt} - \varepsilon \Delta^2 p^{\varepsilon}_{tt} + \Delta p^{\varepsilon} = \operatorname{div} \mathbf{f}, \quad \left. \frac{\partial p^{\varepsilon}_{tt}}{\partial \mathbf{n}} \right|_{\partial\Omega} = 0. \tag{17}$$

Differentiating with respect to t twice the first relation of (17) and denoting $q = p_{tt}$, we find

$$\varepsilon \Delta q^{\varepsilon}_{ttt} - \varepsilon \Delta^2 q^{\varepsilon}_{tt} + \Delta q^{\varepsilon} = \operatorname{div} \mathbf{f}_{tt}, \quad \left. \frac{\partial q^{\varepsilon}}{\partial \mathbf{n}} \right|_{\partial\Omega} = 0. \tag{18}$$

Let $\{\lambda_n, \phi_n\}$ be the eigenpairs of the Laplacian operator with homogeneous Neumann boundary condition, i.e.

$$-\Delta \phi_n = \lambda_n \phi_n, \quad \left. \frac{\partial \phi_n}{\partial \mathbf{n}} \right|_{\partial\Omega} = 0, \tag{19}$$

with $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots + \infty$. We can then expand $\operatorname{div} \mathbf{f}_{tt}(t)$ and $q^{\varepsilon}(t)$ of (18) by using the eigenfunctions

$$\operatorname{div} \mathbf{f}_{tt}(t) = \sum_{n=0}^{\infty} g_n(t) \phi_n, \qquad q^{\varepsilon}(t) = \sum_{n=0}^{\infty} q_n(t) \phi_n. \tag{20}$$

Plugging (20) into (18), we obtain

$$\varepsilon q_n'''(t) + \varepsilon \lambda_n q_n''(t) + q_n(t) = -\frac{1}{\lambda_n} g_n(t), \quad \forall n \geq 1. \tag{21}$$

The characteristic form of this third-order ordinary differential equation is

$$\varepsilon x^3 + \varepsilon \lambda_n x^2 + 1 = 0, \quad \forall n \geq 1. \tag{22}$$

One notes immediately that (22) does not admit any real positve root. On the other hand, the three roots of the above equation, x_1^n, x_2^n, x_3^n, satisfy, in particular, the following relation:

$$\frac{1}{x_1^n} + \frac{1}{x_2^n} + \frac{1}{x_3^n} = 0.$$

Hence, there must be a negative real root and two complex roots with positive real part, i.e.

$$x_1^n = \overline{x_2^n} = a_n + ib_n \quad \text{with } a_n > 0 \text{ and } x_3^n < 0.$$

Therefore, the general solution of (21) is of the form

$$q_n(t) = c_1(t) e^{(a_n + ib_n)t} + c_2(t) e^{(a_n - ib_n)t} + c_3(t) e^{x_3^n t}.$$

Since $a_n > 0$, we conclude that $q_n(t)$ and, therefore, $p_{tt}(t)$ cannot be bounded uniformly for $t \in [\delta, +\infty)$ for any $\delta > 0$.

The above method can be generalized to show that, for any $n \geq 3$, the following system

$$\mathbf{u}_t^\varepsilon - \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, \tag{23}$$

$$\operatorname{div} \mathbf{u}^\varepsilon - \varepsilon \Delta \partial_t^{n-1} p^\varepsilon = 0, \quad \left. \frac{\partial(\partial_t^{n-1} p^\varepsilon)}{\partial \mathbf{n}} \right|_{\partial\Omega} = 0, \tag{24}$$

cannot be used as an approximation to the linearizaed Navier–Stokes equations. Indeed, repeating the process from (18) to (22), we find that the characteristic form of the $(n-1)$th-order ordinary differential equations corresponding to (23) and (24) is

$$\varepsilon x^n + \varepsilon \lambda x^{n-1} + 1 = 0, \tag{25}$$

where $\lambda \geq 0$ is an eigenvalue of the Laplacian operator with homogeneous Neumann boundary condition. We recall that the n roots $x_1, x_2, \ldots, x_n$ of (25) satisfy, in particular, the following relation:

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 0. \tag{26}$$

It is obvious from (25) that none of the x_i's can be positive real number. We then derive from (26) that there exist at least one pair of conjugate complex roots. Hence, we can arrange x_i's as follows:

$$(x_{2j-1}, x_{2j}) = (a_j + \mathrm{i}b_j, a_j - \mathrm{i}b_j), \quad j = 1, 2, \ldots, k \ (k \geq 1)$$

and $x_{2k+1}, x_{2k+2}, \ldots, x_n$ are all negative real numbers. Since

$$\frac{1}{x_{2j-1}} + \frac{1}{x_{2j}} = \frac{2a_j}{a_j^2 + b_j^2},$$

we derive from (26) that

$$0 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \sum_{j=1}^{k} \frac{2a_j}{a_j^2 + b_j^2} + \sum_{j=2k+1}^{n} \frac{1}{x_j}.$$

Hence, at least one of a_j will be positive and, consequently, the solution of (23) and (24) cannot be uniformly bounded for $t \in [\delta, +\infty)$ for any $\delta > 0$.

Remarks

A similar (and simpler) analysis applied to (10–11) and (12–13) shows them both to be stable.

It turns out (P. Gresho, personal communication) that, in an independent effort, James A. Schutt at Sandia National Laboratory has actually tested the projection-3 scheme on the Navier–Stokes equations, and also found it to be unconditionally unstable.

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